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Estimates of the Volume of Solutions of Differential Equations with Hukuhara Derivative

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Abstract—For a class of nonlinear differential equations with Hukuhara derivative, lower bounds for the volume of their solutions are obtained. A. D. Aleksandrov's classical inequalities for mixed volumes combined with the comparison method are used.

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1. INTRODUCTION

Differential equations with Hukuhara derivative and their generalizations, fuzzy differential equations [1], [2] are used in the study of the dynamics of systems under conditions of indeterminacy, ambiguity, and incompleteness of information. The role of this class of differential equations in the study of generalized differential equations was discussed in Tolstonogov's monograph [3]. The comparison method and Lyapunov's direct method were developed in the monograph [4]. It follows from the analysis in this monograph and from many papers concerned with the use of the comparison method and Lyapunov's direct method for this class of equations that the main impediment to the application of these methods is that there are no methods for constructing Lyapunov functions. In the present paper, for a class of nonlinear differential equations with Hukuhara derivative, we propose to use the volume of the solutions of the equations as an analog of the Lyapunov function. Although such a volume does not possess properties typical of the classical Lyapunov function, it is, nevertheless, a natural measure associated with the solutions of this class of equations. Using the powerful geometric apparatus created by H. Minkowski and A. D. Aleksandrov, we can establish lower bounds for the volume of solutions of certain differential equations with Hukuhara derivative.

2. AUXILIARY RESULTS

Let $\mathrm{conv}(\mathbb{R}^n)$ be the metric space of convex compact sets from $\mathbb{R}^n$ with the Hausdorff metric. In the space $\mathrm{conv}(\mathbb{R}^n)$, the operations of (Minkowski) addition and multiplication by a nonnegative scalar are defined. If $A \in \mathfrak{L}(\mathbb{R}^n)$, where $\mathfrak{L}(\mathbb{R}^n)$ is the set of linear operators $\mathbb{R}^n$, then the action of the operator A can be extended in a natural way to the space $\mathrm{conv}(\mathbb{R}^n)$:

$$Au = \{Ax : x \in u\} \in \mathrm{conv}(\mathbb{R}^n), \qquad u \in \mathrm{conv}(\mathbb{R}^n).$$

Let $u \in \mathrm{conv}(\mathbb{R}^n)$, $v \in \mathrm{conv}(\mathbb{R}^n)$. If there exists an element $w \in \mathrm{conv}(\mathbb{R}^n)$ such that $u = w + v$, then the element w is called the *Hukuhara difference of the elements* u *and* v. The notation $w = u - v$ is used. The difference of two elements of the space $\mathrm{conv}(\mathbb{R}^n)$ does not always exist.

The notion of Hukuhara difference allows us to define the notion of Hukuhara derivative for the mapping $F\colon (\alpha, \beta) \to \mathrm{conv}(\mathbb{R}^n)$, $(\alpha, \beta) \subset \mathbb{R}$.

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Definition. A mapping $F\colon (\alpha,\beta)\to \operatorname{conv}(\mathbb{R}^n)$ is said to be *differentiable at a point* $t_0\in(\alpha,\beta)$ if there exists an element $D_HF(t_0)\in\operatorname{conv}(\mathbb{R}^n)$ such that the limits

$$\lim_{\varrho\to 0+}\frac{F(t_0+\varrho)-F(t_0)}{\varrho},\qquad \lim_{\varrho\to 0+}\frac{F(t_0)-F(t_0-\varrho)}{\varrho}$$

exist and are $D_HF(t_0)$. In this case, $D_HF(t_0)$ is called the *Hukuhara derivative at the point* t_0. Differentiability on open, half-open, and closed intervals is defined in the standard way.

A mapping $F(t)$ differentiable on $[a,b]\subset\mathbb{R}$ can be recovered from its derivative by using the Aumann integral [4]

$$F(t)=F(a)+\int_a^t D_HF(s)\,ds,\qquad t\in[a,b].$$

Note that the necessary condition for the differentiability of the mapping is that the function $\operatorname{diam}F(t)$ is nondecreasing.

Following Aleksandrov's papers [5]–[8], let us present some results (necessary for further exposition) from the geometry of convex bodies.

Let $\operatorname{int}u_i\neq\varnothing$, let $V[u_1,u_2,\dots,u_n]$ denote the mixed volume of the convex compact sets u_i, and let $V[u]$ be the volume of the body u,

$$V[u]=V\,[\underbrace{u,u,\dots,u}_{n}],\qquad V_m[u_1,u_2]=V\,[\underbrace{u_1,\dots,u_1,\overbrace{u_2,\dots,u_2}^{m}}_{n}],\quad m=1,\dots,n-1,$$
$$V_0[u_1,u_2]=V[u_1],\qquad V_n[u_1,u_2]=V[u_2]. \tag{2.1}$$

The functional $V[u_1,\dots,u_n]$ is additive and positively homogeneous in each argument, invariant with respect to the rearrangement of the arguments, as well as continuous in all of its arguments with respect to the Hausdorff metrics. Therefore, the Steiner formula is valid:

$$V[u_1+\varrho u_2]=\sum_{k=0}^{n}C_n^k\varrho^kV_k[u_1,u_2],\qquad \varrho\in\mathbb{R}_+. \tag{2.2}$$

It follows from this formula that

$$nV_1[u_1,u_2]=\lim_{\varrho\to0}\frac{V[u_1+\varrho u_2]-V[u_1]}{\varrho}.$$

Aleksandrov's inequality

$$V^2[u_1,\dots,u_{n-1},u_n]\ge V[u_1,\dots,u_{n-2},u_{n-1},u_{n-1}]V[u_1,\dots,u_{n-2},u_n,u_n] \tag{2.3}$$

implies the following inequalities for the functionals $V_k[u_1,u_2]$:

$$V_k^2[u_1,u_2]\ge V_{k-1}[u_1,u_2]V_{k+1}[u_1,u_2],$$

which yield the estimates

$$V_k[u_1,u_2]\ge V^{(n-k)/n}[u_1]V^{k/n}[u_2]. \tag{2.4}$$

The particular case of the last inequality for $k=1$ is the Brunn–Minkowski isoperimetric inequality.

3. STATEMENT OF THE PROBLEM

Consider the following differential equation in the space $\operatorname{conv}\mathbb{R}^n$:

$$D_Hu(t)=\psi(t,V[u(t)])Au(t), \tag{3.1}$$

where D_H is the Hukuhara derivative, $u\in\operatorname{conv}(\mathbb{R}^n)$, $\psi\in C([a,+\infty)\times[b,+\infty);\mathbb{R}_+)$, $b>0$, and $A\in\mathfrak{L}(\mathbb{R}^n)$.

Consider the function $v_0(t) = V[u(t)]$, where $u\colon [t_0, \Omega^+(t_0, u_0)) \to \mathrm{conv}(\mathbb{R}^n)$ is a solution of the Cauchy problem of the differential equation (3.1) with the initial condition $u(t_0) = u_0$, $u_0 \in \mathrm{conv}(\mathbb{R}^n)$, $t_0 > a$, and $V[u_0] \ge b$. Let us consider how the function $v_0(t)$ changes along the solution $u(t)$. It follows from Eq. (3.1) that

$$u(t+\varrho) = u(t) + \int_t^{t+\varrho} \psi(s, V[u(s)])Au(s)\,ds, \qquad \varrho > 0.$$

Using formula (2.2), we obtain

$$\frac{v_0(t+\varrho) - v_0(t)}{\varrho} = \sum_{k=1}^{n} C_n^k \varrho^{k-1} V_k\biggl[u(t), \frac{1}{\varrho}\int_t^{t+\varrho} \psi(s, V[u(s)])Au(s)\,ds\biggr].$$

By the mean-value theorem for the Aumann integral,

$$\lim_{\varrho\to 0+} \frac{1}{\varrho}\int_t^{t+\varrho} \psi(s, V[u(s)])Au(s)\,ds = \psi(t, V[u(t)])Au(t)$$

in the Hausdorff metric [4]; therefore, because the functionals V_k are continuous, we have

$$\lim_{\varrho\to 0+} \frac{v_0(t+\varrho) - v_0(t)}{\varrho} = n\psi(t, v_0(t))V_1[u(t), Au(t)].$$

Similarly, we prove that

$$\lim_{\varrho\to 0-} \frac{v_0(t+\varrho) - v_0(t)}{\varrho} = n\psi(t, v_0(t))V_1[u(t), Au(t)].$$

Therefore,

$$\frac{dv_0(t)}{dt} = n\psi(t, v_0(t))V_1[u(t), Au(t)].$$

Consider the function

$$v_1(t) = V_1[u(t), Au(t)].$$

As above, we can show that

$$\frac{dv_1(t)}{dt} = \psi(t, v_0(t))\bigl((n-1)V_2[u(t), Au(t)] + V_1[u(t), A^2u(t)]\bigr).$$

Denoting $v_2(t) = V_2[u(t), Au(t)]$ and using inequality (2.4) for $k = 1$, we obtain the estimate

$$V_1[u(t), A^2u(t)]) \ge |\det A|^{2/n} V[u(t)],$$

which yields

$$\frac{dv_1(t)}{dt} \ge \psi(t, v_0(t))((n-1)v_2(t) + |\det A|^{2/n} v_0(t)).$$

Similarly, for

$$v_r(t) = V_r[u(t), Au(t)], \qquad r = 2, \dots, n-1,$$

we obtain the representation

$$\frac{dv_r(t)}{dt} = \psi(t, v_0(t))\bigl((n-r)v_{r+1}(t) + rV[\underbrace{u(t), \dots, u(t)}_{n-r}, \underbrace{Au(t), \dots, Au(t)}_{r-1}, A^2u(t)]\bigr). \tag{3.2}$$

Using inequality (2.3), we can write

$$\begin{aligned} &V[\underbrace{u(t), \dots, u(t)}_{n-r}, \underbrace{Au(t), \dots, Au(t)}_{r-1}, A^2u(t)] \\ &\qquad \ge V_r^{(r-1)/r}[u(t), Au(t)] V_r^{1/r}[u(t), A^2u(t)] \ge |\det A|^{2/n} v_r^{(r-1)/r}(t) v_0^{1/r}(t). \end{aligned} \tag{3.3}$$

It follows from (3.2) and (3.3) that

$$\frac{dv_r(t)}{dt} \geq \psi(t, v_0(t))((n-r)v_{r+1}(t) + r|\det A|^{2/n} v_r^{(r-1)/r}(t) v_0^{1/r}(t)). \tag{3.4}$$

The last inequality holds for all $r = 0, 1, \dots, n-1$, and it is also necessary to take into account the equality $v_n(t) = |\det A| v_0(t)$.

Estimate (3.4) is the main estimate, which serves to establish a number of statements about lower bounds for the volume of the solutions of the differential equation (3.1).

4. MAIN RESULT

Consider the differential equation

$$\frac{d\zeta(t)}{dt} = n|\det A|^{1/n} \psi(t, \zeta)\zeta. \tag{4.1}$$

Statement 4.1. *Let the function $v\psi(t, v)$ satisfy a local Lipschitz condition in the variable v on the half-interval $[b, +\infty)$, and let $\zeta\colon [t_0, \omega^+(t_0, \zeta_0)) \to \mathbb{R}_+$ be a solution of the Cauchy problem for Eq.* (4.1) *with the initial condition $\zeta_0 \leq V[u_0]$. Then, for all $t \in [t_0, \min[\Omega^+(t_0, u_0), \omega^+(t_0, \zeta_0)])$, the following estimate holds*:

$$V[u(t)] \geq \zeta(t).$$

Proof. It follows from estimate (3.4) for $r = 0$ and the Brunn–Minkowski inequality that the function $v_0(t)$ satisfies the differential inequality

$$\frac{dv_0(t)}{dt} \geq n|\det A|^{1/n} v_0(t) \psi(t, v_0(t)).$$

Using the differential inequality theorem [9] concludes the proof of the statement. □

Some additional constraints on the function $\psi(t, v)$ allow us to obtain sharper estimates for the volume of the solution of Eq. (3.1). Let k be a natural number, $2 \leq k \leq n-1$. Consider the nonlinear system of differential equations

$$\begin{aligned} \frac{d\theta_r(t)}{dt} &= \psi(t, \theta_0(t))((n-r)\theta_{r+1}(t) + r|\det A|^{2/n} \theta_r^{(r-1)/r}(t) \theta_0^{1/r}(t)), \quad r = 0, \dots, k-1, \\ \frac{d\theta_k(t)}{dt} &= \psi(t, \theta_0(t))((n-k)|\det A|^{(k+1)/n} \theta_0(t) + k|\det A|^{2/n} \theta_k^{(k-1)/k}(t) \theta_0^{1/k}(t)). \end{aligned} \tag{4.2}$$

Statement 4.2. *Let the function $\psi(t, v)$ satisfy a local Lipschitz condition in the variable v on the half-interval $[b, +\infty)$ and be nondecreasing in the variable v, and let the function $\theta\colon [t_0, \omega^+(t_0, \theta_0)) \to \mathbb{R}_+^{k+1}$ be a solution of the Cauchy problem for the system of equations* (4.2) *with the initial conditions*

$$\theta_0(t_0) \leq V[u_0], \quad \theta_r(t_0) \leq V_r[u_0, Au_0], \qquad r = 1, \dots, k.$$

Then, for all $t \in [t_0, \min[\Omega^+(t_0, u_0), \omega^+(t_0, \theta(t_0))])$, the following estimate holds:

$$V[u(t)] \geq \theta_0(t). \tag{4.3}$$

Proof. Applying inequality (2.4) to inequality (3.4) for $r = k$, we obtain the estimate

$$\begin{aligned} \frac{dv_k(t)}{dt} &\geq \psi(t, v_0(t))((n-k)v_{k+1}(t) + k|\det A|^{2/n} v_k^{(k-1)/k}(t) v_0^{1/k}(t)) \\ &\geq \psi(t, v_0(t))((n-k)|\det A|^{(k+1)/n} v_0(t) + k|\det A|^{2/n} v_k^{(k-1)/k}(t) v_0^{1/k}(t)). \end{aligned} \tag{4.4}$$

Since the function $\psi(t, v)$ is nondecreasing in v, the right-hand sides of the system of differential inequalities composed of inequalities (3.4) for $r = 0, \dots, k$ and of inequality (4.4) satisfy the Ważevski condition [9] in the cone $\mathbb{R}_+^k$. Since the solutions of this system are positive under the condition that the initial data are positive, it follows that, by the differential inequality theorem [9], inequality (4.3) holds. The statement is proved. □

Consider the nonlinear system

$$\begin{aligned} \frac{d\eta_r(t)}{dt} &= \psi(t,\eta_0(t))((n-r)\eta_{r+1}(t) + r|\det A|^{(r+1)/n}\eta_0(t)), \qquad r = 0,\dots,k-1, \\ \frac{d\eta_k(t)}{dt} &= n|\det A|^{(k+1)/n}\psi(t,\eta_0(t))\eta_0(t). \end{aligned} \tag{4.5}$$

Statement 4.3. *Let the function $\psi(t,v)$ satisfy a local Lipschitz condition in the variable v on the half-interval $[b,+\infty)$ and be nondecreasing in the variable v, and let the function $\eta\colon [t_0,\omega^+(t_0,\theta_0)) \to \mathbb{R}_+^{k+1}$ be a solution of the Cauchy problem for the system of equations* (4.5) *with the initial conditions*

$$\eta_0(t_0) \le V[u_0], \quad \eta_r(t_0) \le V_r[u_0, Au_0], \qquad r = 1,\dots,k.$$

Then, for all $t \in [t_0, \min[\Omega^+(t_0,u_0), \omega^+(t_0,\eta(t_0))])$, the following estimate holds:

$$V[u(t)] \ge \eta_0(t).$$

Proof. In order to prove this statement, it suffices to apply inequalities (3.1) to inequality (3.4); this yields the estimates

$$\begin{aligned} \frac{dv_r(t)}{dt} &\ge \psi(t,v_0(t))((n-r)v_{r+1}(t) + r|\det A|^{2/n}v_r^{(r-1)/r}(t)v_0^{1/r}(t)) \\ &\ge \psi(t,v_0(t))((n-r)v_{r+1}(t) + r|\det A|^{(r+1)/n}v_0(t)), \qquad r = 0,\dots,k-1, \\ \frac{dv_k(t)}{dt} &\ge \psi(t,v_0(t))((n-k)v_{k+1}(t) + r|\det A|^{2/n}v_k^{(k-1)/k}(t)v_0^{1/k}(t)) \\ &\ge n|\det A|^{(k+1)/n}\psi(t,v_0(t))v_0(t). \end{aligned}$$

In the case $k=1$, system (4.5) takes the form

$$\frac{d\eta_0(t)}{dt} = n\psi(t,\eta_0(t))\eta_1(t), \qquad \frac{d\eta_1(t)}{dt} = n|\det A|^{2/n}\psi(t,\eta_0(t))\eta_0(t). \tag{4.6}$$

In this case, we can weaken the assumptions with respect to the function $\psi(t,v)$. $\square$

Statement 4.4. *Let the function $v\psi(t,v)$ satisfy a local Lipschitz condition in the variable v on the half-interval $[b,+\infty)$ and be nondecreasing in the variable v, and let the function $\eta\colon [t_0,\omega^+(t_0,\eta(t_0))) \to \mathbb{R}_+^2$ be a solution of the Cauchy problem for the system of equations* (4.6) *with the initial conditions*

$$\eta_0(t_0) = V[u_0], \qquad \eta_1(t_0) = V_1[u_0, Au_0].$$

Then, for all $t \in [t_0, \min[\Omega^+(t_0,u_0), \omega^+(t_0,\eta(t_0))])$, the following estimate holds:

$$V[u(t)] \ge \eta_0(t).$$

Proof. Along with system (4.6), we consider the auxiliary system

$$\frac{d\overline{\eta}_0(t)}{dt} = n\psi(t,\overline{\eta}_0(t))\overline{\eta}_1(t) - \varepsilon, \qquad \frac{d\overline{\eta}_1(t)}{dt} = n|\det A|^{2/n}\psi(t,\overline{\eta}_0(t))\overline{\eta}_0(t) - \varepsilon \tag{4.7}$$

with the initial conditions

$$\overline{\eta}_0(t_0,\varepsilon) = V[u_0] - \varepsilon, \qquad \overline{\eta}_1(t_0) = V_1[u_0, Au_0] - \varepsilon,$$

where ε is a sufficiently small positive number. Let

$$m(t) = V[u(t)] - \overline{\eta}_0(t,\varepsilon).$$

Let T be a number such that

$$t_0 < T < \min[\Omega^+(t_0,u_0), \omega^+(t_0,\eta(t_0))].$$

Then there exists a sufficiently small positive number ε_0 such that, for all ε, $0 < \varepsilon < \varepsilon_0$, the solution $\overline{\eta}_0(t,\varepsilon)$, $\overline{\eta}_1(t,\varepsilon)$ exists on the closed interval $[t_0, T]$, is positive, and

$$\overline{\eta}_0(t,\varepsilon) \to \eta_0(t), \quad \overline{\eta}_1(t,\varepsilon) \to \eta_1(t) \qquad \text{as} \quad \varepsilon \to 0+$$

uniformly in $t \in [t_0, T]$. Let us prove that

$$m(t) > 0 \qquad \text{for all} \quad t \in [t_0, T].$$

Indeed, if this is so, then since $m(t_0) > 0$, there exists an instant of time $t_1 \in (t_0, T]$ such that $m(t_1) = 0$ and $m(t) > 0$ for all $t \in [t_0, t_1)$. Therefore, $(dm/dt)(t_1) \le 0$. On the other hand, it follows from inequality (3.4) for $r = 0$ and the first equation of system (4.7) that

$$\frac{dm}{dt}(t_1) = \varepsilon + n\psi(t_1, \overline{\eta}_0(t_1))(v_1(t_1) - \overline{\eta}_1(t_1)).$$

If $v_1(t_1) - \overline{\eta}_1(t_1) \ge 0$, then we immediately obtain a contradiction. But if the inequality

$$v_1(t_1) - \overline{\eta}_1(t_1) < 0$$

holds, then there exists an instant of time $t_2 \in [t_0, t_1)$ such that

$$v_1(t_2) - \overline{\eta}_1(t_2) = 0, \qquad v_1(t) - \overline{\eta}_1(t) > 0, \quad t \in [t_0, t_2).$$

In this case, it follows from inequality (3.4) for $r = 1$ and the second equation of system (4.7) that

$$0 \ge \frac{dv_1}{dt}(t_2) - \frac{d\overline{\eta}_1}{dt}(t_2) \ge \varepsilon + n|\det A|^{2/n}(v_0(t_2)\psi(t_2, v_0(t_2)) - \overline{\eta}_0(t_2)\psi(t_2, \overline{\eta}_0(t_2))) \ge \varepsilon.$$

The resulting contradiction proves the inequality $v_0(t) > \overline{\eta}(t,\varepsilon)$ for $t \in [t_0, T]$. The passage to the limit as $\varepsilon \to 0+$ concludes the proof of the statement. □

Note that the system of equations (4.6) can readily be reduced to one equation

$$\frac{d\eta_0}{dt} = n\psi(t, \eta_0)\sqrt{|\det A|^{2/n}\eta_0^2 + V_1^2[u_0, Au_0] - |\det A|^{2/n}V^2[u_0]}, \qquad \eta_0(t_0) = V[u_0].$$

Example. Consider the differential equation

$$D_H u(t) = Au(t), \tag{4.8}$$

where $u \in \operatorname{conv}(\mathbb{R}^n)$ and $A \in \mathfrak{L}(\mathbb{R}^n)$. Applying Statement 4.1, we obtain the following estimate for the volume of the solution of this equation:

$$V[u(t)] \ge V[u_0]e^{n|\det A|^{1/n}(t-t_0)}, \qquad t \ge t_0. \tag{4.9}$$

Statement 4.4 implies the estimate

$$V[u(t)] \ge V[u_0]\cosh(n|\det A|^{1/n}(t-t_0)) + \frac{V_1[u_0, Au_0]}{|\det A|^{1/n}}\sinh(n|\det A|^{1/n}(t-t_0)), \qquad t \ge t_0. \tag{4.10}$$

Thus, estimate (4.10) is sharper than estimate (4.9). However, in order to apply estimate (4.10), we need to know more about the initial conditions: it is necessary to know not only the volume $V[u_0]$, but also the first mixed volume $V_1[u_0, Au_0]$. For example, if u_0 is the unit ball, then, in addition to its volume, it is necessary to know the area of the surface of the ellipsoid, which is the image of this ball under the linear transformation A. Also note that, for operators close to scalar ones, estimates (4.9) and (4.10) lead to practically the same result. The systems of comparison equations obtained in the present paper for $k \ge 2$ allow us to obtain still sharper estimates of the volume of the solution $u(t)$; however, it is necessary to have even greater knowledge about the initial condition; namely, it is necessary to know all the mixed volumes $V_r[u_0, Au_0]$, $r = 0, \dots, k$.

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